denced by the positive member of the pair of real roots, represents a difficulty in adapting the theory. The conclusion implies that the unspecified initial condition $\lambda_y^*(0)$ must be chosen to suppress the unstable component of the solution, if this is possible, and this is closely related to the "matching" problem of matched asymptotic expansions.

If one attempts to proceed with expansion when the roots are imaginary, oscillations are encountered in the boundary layer which cannot be suppressed for large τ by choice of a single initial value, and the procedure fails. The borderline case of repeated zero roots encounters secular growth, corresponding also to failure, and hence it appears that the strengthened Legendre-Clebsch condition at the endpoints is essential for success.

In the terminal boundary layer, stability for reversed time is required by the Tihonov theory but, again, one or the other of the pair of real roots will obviate it, and suppression of instability by the choice of an end condition will again be needed. Zero or imaginary roots again signal failure.

After calculating the solution of the reduced system, and then the initial and terminal boundary layer solutions, each in turn, and combining them à la Vasil'eva (Theorem 40.1 of Ref. 1), one then has an approximation to the solution. Higher-order approximations are also offered by the Vasil'eva theory. The value of approximations of various order in flight mechanics applications, such as turn and climb performance of aircraft,[2] remains to be assessed.

Example

The following simple example is of interest. Take $f = \frac{1}{2}(Ay^2 + Bu^2)$, $g = u$, and end conditions $y(0) = y(t_f) = 1$, $x(0) = 0$. The solution of the reduced problem is $\bar{u} = \bar{y} = \bar{x} = \bar{\lambda}_y = 0$, $\bar{\lambda}_x = -1$. The solution of the initial boundary-layer equations takes the form

$$y^*(\tau) = Ce^{\gamma\tau} + De^{-\gamma\tau} \tag{25}$$

where $\gamma \equiv (A/B)^{1/2}$ and the constants C and D must be chosen to satisfy $y^*(0) = 1$, and $y^* \to \bar{y} = 0$ for large τ. This illustrates the selection of the undetermined multiplier initial value (buried in C and D) to suppress instability in the boundary layer, i.e., $C = 0$, $D = 1$. The situation in the terminal layer is similar, except that the "stable" component is suppressed. The composite solution, representing an expansion to zero-order in ϵ, is

$$\hat{y}(t) = e^{-\gamma\tau} + e^{\gamma(t-t_f)} \tag{26}$$

which may be compared with the exact solution

$$y = [(1 - e^{-\gamma t_f})e^{\gamma t} + (e^{\gamma t_f} - 1)e^{-\gamma t}]/(e^{\gamma t_f} - e^{-\gamma t_f}) \tag{27}$$

The approximation appears to be good for t_f large and/or γ large.

Concluding Remarks

The preceding analysis of a simplified two-state-variable case suggests that, when the Legendre-Clebsch condition for the reduced problem is met in strengthened form at the endpoints, initial values for boundary-layer equations should be chosen to suppress unstable solution components, if possible. Further work is needed to explore the possibility of suppression. The appearance of singular points and/or arcs in the interior of reduced system solutions is also of interest. Extensions of the analysis to more general problems, and particularly to those with more state variables, are obviously needed.

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Linear Instability of Far-Wake Flow

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Nomenclature

b = wake half-width defined as width from wake axis to point where $\Delta u = 0.5\Delta u_{₵}$ (physical coordinates)
$\bar{b}$ = wake half width in Howarth coordinates
c = wave velocity ($= c_R + ic_I$)
c_I = dimensional amplification factor
d = cylinder diameter
$\tilde{e}_f$ = rms hot wire fluctuation voltage at frequency f
M_e = wake edge Mach number
ΔM = relative Mach number [$= (u_e - u_{₵})/a_e$]
$\tilde{m}'$ = rms mass flow fluctuation
Re_d = freestream Reynolds number based on cylinder diameter
ΔT = temperature excess [$= (T_{₵} - T_e)/T_e$]
x = axial distance from cylinder
y = normal distance from wake axis
α = dimensional wave number

Subscripts

e = wake edge value
₵ = wake centerline value

Introduction

MEASUREMENTS of the mean wake flow behind circular cylinders at Mach 6, described in Ref. 1, showed that in a certain Reynolds number range ($Re_d \simeq 300$–4000) the inner wake stemming from the body boundary layers is laminar and loses its identity within 60 diam, and the outer wake, stemming from the bow shock, is also laminar within a certain downstream distance, and may be calculated from a laminar linear theory, but does not become similar (gaussian) within 2400 diam. Experimental study of the stability of these wakes is presented in Ref. 2.

In Ref. 2, the amplification rates of fluctuations measured in the regimes where the mean flowfield is predicted by laminar two-dimensional theory were compared with amplification rates calculated from the linear two-dimensional stability theory of compressible wakes by Lees and Gold.[3] In the theory, mean profiles of velocity and temperature were assumed to be gaussian distribution in the Howarth-transformed variable. The most unstable frequencies were predicted quite well but the measured rates of amplifications were somewhat lower than the theoretical values (see Fig. 10, Ref. 2). At the time, it was suggested that the discrepancy between the experimental results and stability theory was caused by the discrepancy in the mean wake profile shape.

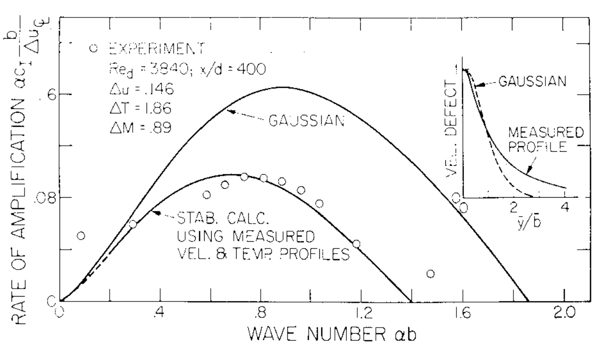

Fig. 1 Experimental and theoretical amplification rates at $Re_d = 3840$; $x/d = 400$.

Received March 5, 1970. This work was carried out under the sponsorship and with the financial support of the U.S. Army Research Office and the Advanced Research Projects Agency under Contract DA-31-124-ARO(D)-33, part of Project DEFENDER sponsored by the Advanced Research Projects Agency.

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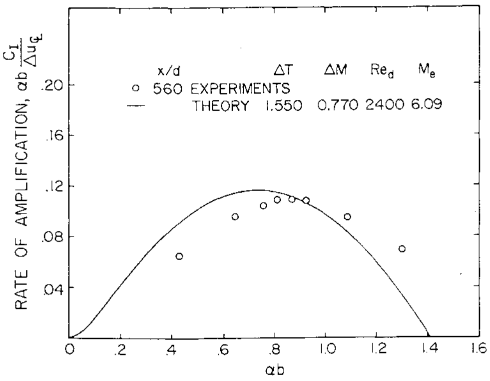

Fig. 2 Experimental and theoretical amplification rates at $Re_d = 2400$; $x/d = 560$.

Mack[4] has studied very extensively linear stability theory of laminar compressible boundary layers by integrating the Orr-Summerfeld equations on an electronic digital computer. His program for inviscid stability computation was adapted to calculate the eigenfunctions and eigenvalues of the stability equations for arbitrary wake profiles by changing the boundary condition at the wake axis. The antisymmetrical mode as defined by Lees and Gold[3] is the most unstable mode $[v'(0) \neq 0, u'(0) = 0]$ and is used here in the wake stability calculations.

Comparison between Experiment and Stability Theory

Amplification rates

For three different cases, a comparison is made between the theoretical predictions and the experimentally determined amplification rates (Figs. 1–3). In the theory the measured mean profiles were used, and the results compare favorably with the measured rates. In Fig. 1, the previously calculated amplification curve for a gaussian profile is shown also, together with the gaussian and measured velocity profiles. These results show that the mean profile shapes are very important for the amplification rates of infinitesimal fluctuations.

Distributions of mass flow fluctuations across the wake

As pointed out in Ref. 2, the hot wire with large overheats measures nearly the mass-flow fluctuations. Therefore, in the region where the linear stability theory is applicable, the distribution of the hot wire fluctuation voltage should be similar to that of the mass-flow fluctuations as calculated from stability theory.

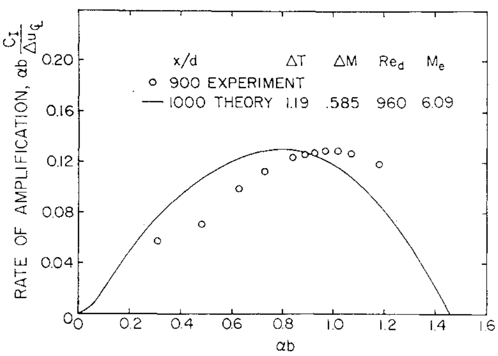

Fig. 3 Experimental and theoretical amplification rates at $Re_d = 960$; $x/d = 900$.

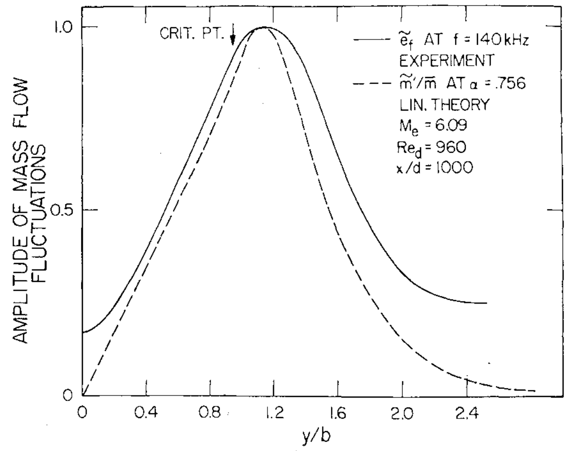

Fig. 4 Amplitude distributions of mass-flow fluctuations across wake at $f = 140$ kHz.

In Figs. 4 and 5, two examples are shown for comparison. At the lower wave number (corresponding to a lower frequency) the comparison of the measured voltage fluctuation distribution and the calculated mass-flow fluctuation is quite good (Fig. 4). At the higher frequency (Fig. 5) the location of maximum oscillation compares well. (Note that the location of maximum oscillation moves towards the wake axis with increasing frequency.)

However, at this higher frequency the theoretical mass-flow fluctuation distribution is narrower than the measured distribution. The theoretical distribution of fluctuations shows a rather sharp peak at the maximum value as shown in Fig. 5. In the neighborhood of this point viscous effects must be included, which would smooth out the fluctuation profile near the maximum.

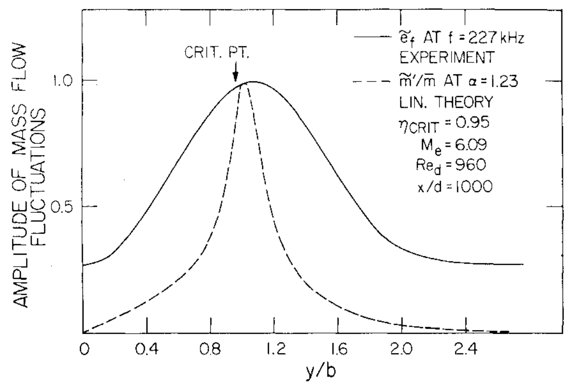

Fig. 5 Amplitude distributions of mass-flow fluctuations across wake at $f = 227$ kHz.

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